4/19/2001 MA401 Complex Analysis Name:_Solution__________
Dr. Lunsford Test 2 (100 Points Total)

I. Find all values of $(-4+4\sqrt{3}\,i)^{1/3}$ in $a+bi$ form (round a and b to two decimal places please). Graph these roots on the axes provided below. (10 points)

$(-4+4\sqrt{3}\,i)^{1/3} = (8e^{i(2p/3+2pn)})^{1/3}, n = 0,1,2$, thus the roots are
$1.53+1.29i \quad (n=0)$, $-1.88+.68i \quad (n=1)$, and $.35-1.97i \quad (n=2)$

II. Given the complex function $f(z) = z\cos\left(\frac{1}{z}\right)$, please answer the following. (10 points total)

a. Find the Laurent series expansion for f about zero. (7 points)

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{1-2n}}{(2n)!} = z - \frac{z^{-1}}{2} + \frac{z^{-3}}{4!} - \frac{z^{-5}}{6!} + \ldots$$

b. Indicate the values of c_0, c_1, and $\mathrm{Re}\,s(f,0)$ from the Laurent series found in part (a) above. (3 points)

$c_0 = 0$, $c_1 = 1$, $\mathrm{Re}\,s(f,0) = c_{-1} = -1/2$

III. <u>Without computing the values</u>, plot the 6th roots of unity on the complex plane below. Clearly identify the roots (you may call them r_1, r_2, etc.) and show the angle q between the roots. (6 points)

Your picture should show the six roots of unity evenly spaced at angles of 60° apart on the unit circle starting at $z = 1$.

IV. Given the function $f(z) = \dfrac{\sin(z)}{z^2(z-i)^2}$, please answer the following. (10 points total)

a. Find the order of the pole at $z = i$ (you must justify your answer). (5 points)

$\lim\limits_{z\to i}(z-i)^2 \dfrac{\sin z}{z^2(z-i)^2} = -\sin i \neq 0$, therefore the order of the pole at $z = i$ is two.

b. Find $\mathrm{Re}\,s(f,0)$. (5 points)

First note that f has a pole of order one at zero since $\lim_{z\to 0} z\dfrac{\sin z}{z^2(z-i)^2} = -1 \neq 0$.
Since the pole is of order one this same limit also give the residue. Hence
$\operatorname{Re}s(f,0) = -1$.

V. Evaluate the integrals over the indicated curves. For each problem clearly indicate what theorem(s) you use and why you know you may use the theorem(s). Please draw pictures to help clarify your ideas!! (8 points each – 64 points total)

a. $\oint_\Gamma \dfrac{z^3}{z^2+1}dz$, where Γ is the closed curve given by $|z-i| = 1/4$.

First note that the closed curve Γ only contains the zero $z = i$ of z^2+1. We can use either the Residue Theorem or Cauchy's Integral Formula:

Residue Theorem: $\oint_\Gamma \dfrac{z^3}{z^2+1}dz = 2p\,i\operatorname{Re}s(\dfrac{z^3}{z^2+1},0) = 2p\,i(-1/2) = -p\,i$

Cauchy's Integral Formula: $\oint_\Gamma \dfrac{z^3}{z^2+1}dz = \oint_\Gamma \dfrac{\left(\dfrac{z^3}{z+i}\right)}{z-i}dz = 2p\,i\left(\dfrac{i^3}{i+i}\right) = -p\,i$

b. $\int_\Gamma \dfrac{1}{z^2+2z+2}dz$, where Γ is the closed curve given by $|z| = 3$.

The zeros of the polynomial z^2+2z+2 are $-1\pm i$. Both of these zeros are in the interior of the closed curve. We can use the Residue Theorem (or the Extended Deformation Theorem along with the C.I.F). I will only show the use of the Residue Theorem here:

$$\int_\Gamma \frac{1}{z^2+2z+2}dz = 2p\,i(\operatorname{Re}s(\frac{1}{z^2+2z+2},-1+i) + \operatorname{Re}s(\frac{1}{z^2+2z+2},-1-i))$$

Now,

$$\operatorname{Re}s(\frac{1}{z^2+2z+2},-1-i) = \lim_{z\to -1-i}\frac{(z-(-1-i))}{(z-(-1-i))(z-(-1+i))} = \lim_{z\to 0}\frac{1}{(z-(-1+i))}$$

$= \dfrac{1}{(-1-i)-(-1+i)} = \dfrac{1}{-2i}$ and similarly $\operatorname{Re}s(\dfrac{1}{z^2+2z+2},-1+i) = \dfrac{1}{2i}$. Thus

$$\int_\Gamma \frac{1}{z^2+2z+2}dz = 2p\,i(1/2i - 1/2i) = 0$$

c. $\int_{\Gamma} \cos(6z)dz$, where Γ is the straight line segment from i to $-i$.

Since $\cos(6z)$ has a continuous antiderivative on $\mathbb{C}$ then

$$\int_{\Gamma} \cos(6z)dz = \int_{i}^{-i} \cos(6z)dz = \frac{1}{6}(\sin(-6i) - \sin(6i)) = \frac{-(\sinh 6)i}{3}$$

d. $\int_{\Gamma} \frac{z^2 + i}{\sin z} dz$, where Γ is the closed curve given by $|z| = 1$.

The closed curve contains only the zero $z = 0$ of $\sin z$ (Note: $\sin z$ has infinitely many zeros!). We cannot use the C.I.F. here, we must use the Residue Theorem:

$$\int_{\Gamma} \frac{z^2 + i}{\sin z} dz = 2p\, i(\mathrm{Re}\, s(\frac{z^2 + i}{\sin z}), 0) = 2p\, i(i) = -2p$$

Note that since $z = 0$ is a zero of order one (i.e. the integrand has a pole of order one at zero), we have that $\mathrm{Re}\, s(\frac{z^2 + i}{\sin z}, 0) = i$.

e. $\int_{\Gamma} \cos\left(\frac{1}{z}\right) dz$, where Γ is the closed curve given by $|z - 1 + 2i| = 1$.

The function $\cos\left(\frac{1}{z}\right)$ is differentiable everywhere but zero. However, zero is not contained in the interior of the closed curve. Thus we can find a simply connected open set on which the function is differentiable and that contains the closed curve. Thus by Cauchy's Integral Theorem this integral is zero.

f. $\int_{\Gamma} \left(\bar{z}\right)^2 dz$, where Γ is the circle of radius 2 traversed once about the point i.

We will need to resort to using the definition of complex integration. First we parameterize the closed curve as follows: $\Gamma(t) = i + 2e^{it}, 0 \le t \le 2p$. Next we note that $\overline{\Gamma(t)} = \overline{i + 2e^{it}} = -i + 2e^{-it}$ and $\Gamma'(t) = 2ie^{it}dt$ which gives

$$\int_{\Gamma} \left(\bar{z}\right)^2 dz = \int_{0}^{2p} (-i + 2e^{-it})^2 2ie^{it} dt = 2i \int_{0}^{2p} (-e^{it} - 4i + 4e^{-it}) dt = 16p$$

g. $\int \frac{\cos(z^2)}{(z - \sqrt{p}\, i)^3} dz$, where Γ is the closed curve given by $|z| = 3$.

We will use the C.I.F. for higher order derivatives (you can also use the Residue Theorem but be careful to note that the pole at $\sqrt{p}\, i$ is of order 3).

$$\int \frac{\cos(z^2)}{(z-\sqrt{p}\, i)^3} dz = \frac{2p\, i}{2!} f^{(2)}(\sqrt{p}\, i) \text{ where } f(z) = \cos(z^2).$$

Now, $f^{(2)}(z) = -2\sin(z^2) - 4z^2\cos(z^2)$ and $f^{(2)}(\sqrt{p}\, i) = -4p$

Thus $\int \frac{\cos(z^2)}{(z-\sqrt{p}\, i)^3} dz = -4p^2 i$

h. $\oint_\Gamma \frac{3+z^2}{(z+i)^2 z} dz$, where Γ is the circle of radius 2 traversed once about the origin.

This function has two poles, one of order one at zero and one of order two at $-i$. By the Residue Theorem we have

$$\oint_\Gamma \frac{3+z^2}{(z+i)^2 z} dz = 2p\, i(\mathrm{Re}\, s(\frac{3+z^2}{(z+i)^2 z}, -i) + \mathrm{Re}\, s(\frac{3+z^2}{(z+i)^2 z}, 0))$$

Since the pole at zero is of order one we will compute the residue there first:

$$\mathrm{Re}\, s(\frac{3+z^2}{(z+i)^2 z}, 0) = \lim_{z\to 0} \frac{3+z^2}{(z+i)^2} = -3$$

and since the pole at $-i$ is of order two we have:

$$\mathrm{Re}\, s(\frac{3+z^2}{(z+i)^2 z}, -i) = \frac{1}{1!} \lim_{z\to -i}(\frac{d}{dz}(3z^{-1} + z)) = \lim_{z\to -i}(-3z^{-2} + 1) = 4$$

Thus $\oint_\Gamma \frac{3+z^2}{(z+i)^2 z} dz = 2p\, i(4-3) = 2p\, i$